

Charged BTZ-like Black Holes in Higher Dimensions

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Motivated by many worthwhile paper about $(2+1)$ -dimensional BTZ black holes, we generalize them to $(n+1)$ -dimensional solutions, so called BTZ-like solutions. We show that the electric field of BTZ-like solutions is the same as $(2+1)$ -dimensional BTZ black holes, and also their lapse functions are approximately the same, too. By these similarities, it is also interesting to investigate the geometric and thermodynamics properties of the BTZ-like solutions. We find that, depending on the metric parameters, the BTZ-like solutions may be interpreted as black hole solutions with inner (Cauchy) and outer (event) horizons, an extreme black hole or naked singularity. Then, we calculate thermodynamics quantities and conserved quantities, and show that they satisfy the first law of thermodynamics. Finally, we perform a stability analysis in the canonical ensemble and show that the BTZ-like solutions are stable in the whole phase space.

The discovery and investigation of the $(2+1)$ -dimensional BTZ (Banados-Teitelboim-Zanelli) black holes [1–3] organizes one of the great advances in gravity because they provide a simplified model for exploring some conceptual issues, not only about realization of the black hole thermodynamics [4–6] but also about developments in quantum gravity, string and gauge theory, and specially in the context of the AdS/CFT conjecture [7, 8].

The BTZ black hole is suitable, in a nice way, for the AdS/CFT framework and perform a central role in recent investigations and improve our comprehension of low dimensional gravity and of general feature of the gravitational interaction [9].

Interest in the BTZ black hole has recently heightened with the discovery that the thermodynamics of higher-dimensional black holes (see for e.g. [10–12]) can often be understood in terms of the BTZ solution. The entropy of the BTZ black hole as we consider it here, grown from the papers [13, 14] in the case of AdS₃ (and also see [15]). Some class of higher dimensional black holes contains the BTZ black holes in the near-horizon region have been studied in [16, 17]. Also, BTZ-like black holes in even dimensional Lovelock gravity has been investigated in [18] and some solutions of the BTZ black hole in every dimension by defining the singularity as the closed orbits have been discussed in [19].

In recent years there has been increasing interest about black hole solutions whose matter source is power Maxwell invariant, i.e., $(F_{\mu\nu}F^{\mu\nu})^s$ [11]. This theory is considerably richer than that of the linear electromagnetic field and in the special case ($s = 1$) it can reduce to linear field. In addition, in $(n+1)$ -dimensional gravity, for the special choice $s = (n+1)/4$, matter source yields a traceless Maxwell's energy-momentum tensor which leads to conformal invariance. The idea is to take advantage of the conformal symmetry to construct the analogues

of the four dimensional Reissner-Nordström solutions in higher dimensions [12]. Also, it is valuable to find and analyze the effects of exponent s on the behavior of the new solutions, when $s = n/2$. In this case the solutions are completely different from another cases ($s \neq n/2$).

One of the important defects of higher dimensional solutions in Einstein-Maxwell gravity is that in 3-dimension, these solutions do not reduce to BTZ black hole. The aim of this paper is to consider a class of nonlinear electrodynamics field coupled to Einstein gravity and introduce higher dimensional charged BTZ-like black hole. In this solution, the lapse function has a logarithmic term and also the electromagnetic field is proportional to r^{-1} .

Field Equations of Einstein Gravity with nonlinear Electromagnetic Source

The $(n+1)$ -dimensional action in which gravity is coupled to nonlinear electrodynamics field is given by

$$\mathcal{I} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} (\mathcal{R} - 2\Lambda - (\alpha\mathcal{F})^s), \quad (1)$$

where \mathcal{R} is scalar curvature, Λ refers to the negative cosmological constant which is in general equal to $-n(n-1)/2l^2$ for asymptotically AdS solutions, in which l is a scale length factor, α is a constant in which we should fix it and s is the power of nonlinearity and hereafter we set it to $n/2$ to attain BTZ-like solutions. Varying the action (1) with respect to the metric $g_{\mu\nu}$ and the gauge field A_μ , (with $s = n/2$) the field equations are obtained as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\mathcal{R} - 2\Lambda) = \alpha(\alpha\mathcal{F})^{n/2-1} \left(\frac{1}{2}g_{\mu\nu}\mathcal{F} - nF_{\mu\lambda}F_{\nu}{}^{\lambda} \right), \quad (2)$$

$$\partial_\mu \left(\sqrt{-g} F^{\mu\nu} (\alpha\mathcal{F})^{n/2-1} \right) = 0. \quad (3)$$

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Now, we should fix the sign of the constant α in order to ensure the real solutions. It is easy to show that for static diagonal metric in which the nonzero component of A_μ is A_0 , we have

$$\mathcal{F} = F_{\mu\nu}F^{\mu\nu} = -2 \left(\frac{dA_0}{dr} \right)^2,$$

and so the power Maxwell invariant, $(\alpha\mathcal{F})^{n/2}$, may be imaginary for positive α , when $n/2$ is fractional (for even dimension). Therefore, we set $\alpha = -1$, to have real solutions without loss of generality.

(2+1)-dimensional charged BTZ solution

The field equations of three dimensional solution is the same as equations (2) and (3), when we set $n = 2$. The charged BTZ black hole is a solution of the $(2 + 1)$ -dimensional Einstein-Maxwell gravity with a negative cosmological constant $\Lambda = -1/l^2$. The metric is given by [1]

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2d\varphi^2, \quad (4)$$

Here, we use the gauge potential ansatz

$$A_\mu = h(r)\delta_\mu^0 \quad (5)$$

in the electromagnetic field equation (3). We obtain

$$rh''(r) + h'(r) = 0, \quad (6)$$

where prime and double prime are first and second derivative with respect to r , respectively. One can show that the solution of Eq. (6) is $h(r) = q \ln(\frac{r}{l})$ and the electric field in $(2 + 1)$ -dimension is given by

$$F_{tr} = \frac{q}{r}. \quad (7)$$

To find the metric function of (4), one may use any components of Eq. (2) for $n = 2$. Considering the function $h(r)$, the solution of Eq. (2) can be written as

$$N^2(r) = \frac{r^2}{l^2} - \left[M + 2q^2 \ln\left(\frac{r}{l}\right) \right] \quad (8)$$

where $N^2(r)$ is known as the lapse function and M and q are the mass and electric charge of the BTZ black hole, respectively.

$(n + 1)$ -dimensional charged BTZ-like solutions

Here we want to obtain the $(n + 1)$ -dimensional static solutions of Eqs. (2) and (3). We assume that the metric has the following form

$$ds^2 = -F^2(r)dt^2 + \frac{dr^2}{F^2(r)} + r^2 \sum_{i=1}^{n-1} d\phi_i^2, \quad (9)$$

Again, we use the gauge potential like Eq. (5) in nonlinear Maxwell equation (3) for arbitrary n . We obtain surprisingly

$$rh''(r) + h'(r) = 0,$$

with the solution $h(r) = q \ln(\frac{r}{l})$ and so the electric field is the same as $(2 + 1)$ -dimensional BTZ solution (7). It is notable that for higher dimensional linear Maxwell field equation, the electric field depends on the dimensionality but in our case (nonlinear Maxwell field), the electric field is proportional to r^{-1} for arbitrary dimensions.

In order to find the higher dimensional lapse function $F^2(r)$, we should use the Eq. (2). It is easy to show that the solutions of all components of Eq. (2) can be written as

$$F^2(r) = \frac{r^2}{l^2} - r^{2-n} \left[M + 2^{n/2} q^n \ln\left(\frac{r}{l}\right) \right], \quad (10)$$

where M and q are the integration constants which are related to mass and charge parameters, respectively. As one can verify the metric function $F^2(r)$, presented here, differ from the linear higher dimensional Reissner-Nordström black hole solutions; it is notable that the electric charge term in the linear case is proportional to $r^{-(n-2)}$, but in the presented metric function, nonlinear case, this term is logarithmic. In the 3-dimensional limit ($n = 2$), these solutions reduce to the known BTZ solution (8), and because of some similarities in electromagnetic field and metric functions, hereafter we called the higher dimensional solutions as BTZ-like solution.

Properties of the solutions

In order to study the general structure of these space-time, we first investigate the effects of the nonlinearity on the asymptotic behavior of the solutions. It is worthwhile to mention that in arbitrary dimensions, the asymptotic dominant term of Eq. (10) is first term and so BTZ-like solutions are asymptotically AdS.

Second, we look for the essential singularities. After some algebraic manipulation, one can show that for $(n + 1)$ -dimensional charged BTZ-like solutions, the Ricci and Kretschmann scalars are

$$\begin{aligned} R &= -F''(r) - \frac{2(n-1)F'(r)}{r} - \frac{(n-1)(n-2)F(r)}{r^2} \\ &= \frac{-n(n-1)}{l^2} + \frac{2^{n/2}q^n}{r^n}, \end{aligned} \quad (11)$$

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= F'^2(r) + \frac{2(n-1)F'^2(r)}{r^2} \\ &\quad + \frac{2(n-1)(n-2)F^2(r)}{r^4}. \end{aligned} \quad (12)$$

Also one can show that other curvature invariants (such as Ricci square, Weyl square and so on) are functions

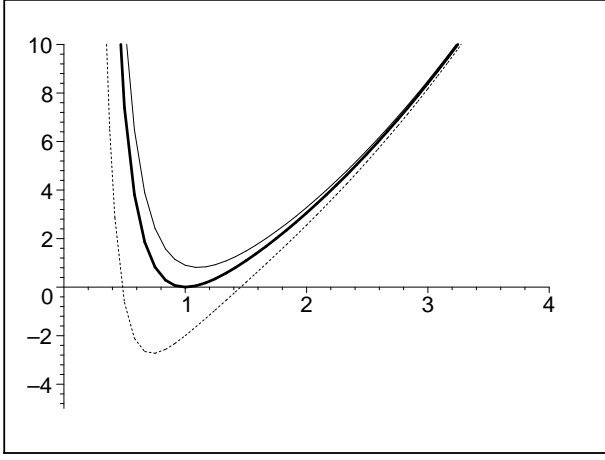


FIG. 1: The lapse function, $F^2(r)$, of BTZ-like solutions versus r for $n = 4$, $l = 1$, $q = 1$, and $M = 0.1$ (no real solution or naked singularity: continuous line), $M = 1$ ($r_- = r_+ = 1$ or extreme black hole: bold line), and $M = 3$ ($r_- = 0.48$, $r_+ = 1.46$ or two horizons: dashed line).

of F'' , F'/r and F/r^2 and therefore it is sufficient to study the Kretschmann scalar for the investigation of the spacetime curvature singularity(ies).

Straightforward calculations show that presented Ricci and Kretschmann scalars with metric function (10) diverges at $r = 0$, is finite for $r \neq 0$ and for large values of r , we have

$$\lim_{r \rightarrow \infty} R = \frac{-n(n-1)}{l^2}, \quad (13)$$

$$\lim_{r \rightarrow \infty} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{2n(n+1)}{l^4}, \quad (14)$$

Equations (13) and (14) confirm that the asymptotic behavior of the solutions are AdS and Also we find that there is a curvature timelike singularity located at $r = 0$.

It is easy to find that the presented black holes (BTZ and BTZ-like) have two inner and outer horizons. For $(n+1)$ -dimensional BTZ-like solutions with arbitrary n , the horizons are located at

$$r_- = l \exp \left\{ -\frac{1}{n} L_W \left(\frac{-nl^{n-2} e^{\left(\frac{-nM}{2^{n/2} q^n} \right)}}{2^{n/2} q^n} \right) - \frac{M}{2^{n/2} q^n} \right\},$$

$$r_+ = l \exp \left\{ -\frac{1}{n} L_W \left(-1, \frac{-nl^{n-2} e^{\left(\frac{-nM}{2^{n/2} q^n} \right)}}{2^{n/2} q^n} \right) - \frac{M}{2^{n/2} q^n} \right\},$$

where the L_W is *LambertW* function satisfies $LambertW(x) \exp[LambertW(x)] = x$ (for more details, see [20] and figures 1). The event horizon is the hypersurface in which light can no longer escape from the gravitational pull of a black hole. For calculation of the event horizon, one can use the time dilation interpretation (gravitational red shift) [21]. It is straightforward

to show that the event horizon of the presented BTZ-like solutions are located at the root(s) of $F^2(r) = 0$. Thus, the presented r_+ is the radius of the event horizon. In addition, Fig.1 shows that depending on the metric parameters, the BTZ-like solutions may be interpreted as black hole solutions with inner and outer horizons, an extreme black hole or naked singularity.

Conserved and thermodynamics quantities

In order to calculate the temperature, one may use of the definition of surface gravity,

$$T_+ = \beta_+^{-1} = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu)}$$

where $\chi = \partial/\partial t$ is the Killing vector. One obtains

$$T_+ = \frac{r_+}{4\pi} \left(\frac{n}{l^2} - \frac{2^{n/2} q^n}{r_+^n} \right). \quad (15)$$

The electric potential U , measured at infinity with respect to the horizon, is defined by [22]

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+} = -q \ln\left(\frac{r_+}{l}\right). \quad (16)$$

More than thirty years ago, Bekenstein argued that the entropy of a black hole is a linear function of the area of its event horizon, which so-called area law [23]. Since the area law of the entropy is universal, and applies to all kinds of black holes in Einstein gravity [23, 24], therefore the entropy of the BTZ-like black holes is equal to one-quarter of the area of the horizon, i.e.,

$$S = \frac{(2\pi r_+)^{n-1}}{4}. \quad (17)$$

The electric charge of the black holes, Q , can be found by calculating the flux of the electromagnetic field at infinity, yielding

$$Q = (2\pi)^{n-2} 2^{(n-6)/2} n q^{n-1}. \quad (18)$$

The present spacetime (9), have boundaries with timelike ($\xi = \partial/\partial t$) Killing vector field. It is straightforward to show that for the quasi local mass we have

$$\mathcal{M} = \int_{\mathcal{B}} d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \xi^b = \frac{(2\pi)^{n-2} (n-1) M}{8}, \quad (19)$$

provided the hypersurface \mathcal{B} contains the orbits of ξ .

Here, we check the first law of thermodynamics for our solutions. We obtain the mass as a function of the extensive quantities S and Q (see appendix for more details). One may then regard the parameters S , and Q as a complete set of extensive parameters for the mass $\mathcal{M}(S, Q)$ and define the intensive parameters conjugate

to them. These quantities are the temperature and the electric potential

$$T = \left(\frac{\partial \mathcal{M}}{\partial S} \right)_Q, \quad U = \left(\frac{\partial \mathcal{M}}{\partial Q} \right)_S \quad (20)$$

It is a matter of straightforward calculation to show that the intensive quantities calculated by Eq. (20) coincide with Eqs. (15) and (16) (see appendix for more details). Thus, these quantities satisfy the first law of thermodynamics

$$d\mathcal{M} = TdS + UdQ.$$

Stability in the canonical Ensemble

Finally, we investigate the stability of charged BTZ-like black hole solutions. The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates, can be studied by the behavior of the energy $\mathcal{M}(S, Q)$ which should be a convex function of its extensive variable. In the canonical ensemble, the charge is fixed parameter, and therefore the positivity of the heat capacity $C_Q = T_+ / (\partial^2 \mathcal{M} / \partial S^2)_Q$ is sufficient to ensure the local stability. Considering the appendix, we can find that $(\partial^2 \mathcal{M} / \partial S^2)_Q$ is

$$\begin{aligned} \left(\frac{\partial^2 \mathcal{M}}{\partial S^2} \right)_Q &= \frac{n}{32\pi^2} \left[\frac{4^{\frac{n}{n-1}} S^{\frac{2-n}{n-1}}}{(n-1)l^2} + \frac{\left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}}}{nS^2} \right] \\ &= \frac{2r_+^{2(1-n)}}{(2\pi)^n} \left(\frac{nr_+^n}{(n-1)l^2} + 2^{n/2} q^n \right). \end{aligned} \quad (21)$$

It is clear that $(\partial^2 \mathcal{M} / \partial S^2)_Q$ is positive and so the heat capacity is always positive for $r \geq r_{ext}$, where the temperature is positive (since for $r \geq r_{ext}$, $F^2(r)$ is an increasing function). Thus, the black hole is stable in the canonical ensemble. It is notable that r_{ext} is the root of temperature, given by

$$r_{ext} = \left(\frac{l^2}{n} \right)^{1/n} \sqrt{2}q. \quad (22)$$

CLOSING REMARKS

One of the significant defects of higher dimensional Reissner-Nordström solutions is that in 3-dimension, these solutions does not reduce to BTZ black hole. In this paper, we have been introduced an action of Einstein-nonlinear Maxwell gravity in which its solutions are very like to BTZ black hole and we have been called them as BTZ-like solutions. It is interesting that, in all dimensions, the electric field of the solutions is proportional to

r^{-1} . Also, it is notable that the lapse function of BTZ-like solutions is very similar to BTZ black hole and its charge term is logarithmic. After these motivations, we found that BTZ-like solutions have a curvature singularity with two horizons, generally. Then, we calculated thermodynamics and conserved quantities and showed that these quantities satisfied the first law of thermodynamics. Finally, we calculated the heat capacity of the BTZ-like black hole solutions and found that they are positive for all the phase space, which means that the black hole is stable for all the allowed values of the metric parameters. This phase behavior is commensurate with the fact that there is no Hawking–Page transition for a black object whose horizon is diffeomorphic to \mathbb{R}^p and therefore the system is always in the high temperature phase.

Finally, it is worthwhile to generalize our static solutions to rotating BTZ-like black hole and it is left for the future.

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APPENDIX

In order to check the first law of thermodynamics, we rewrite the quasilocal mass with respect to the entropy and charge. Considering Eqs. (10), (17) and (18), we have

$$\begin{aligned} \mathcal{M} &= \frac{(n-1)}{32\pi^2} \left[\frac{(4S)^{\frac{n}{n-1}}}{l^2} - \left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}} \ln \left(\frac{(4S)^{\frac{1}{n-1}}}{2\pi l} \right) \right] \\ d\mathcal{M} &= \frac{n}{32\pi^2} \left[\frac{4^{\frac{n}{n-1}} S^{\frac{1}{n-1}}}{l^2} - \frac{\left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}}}{nS} \right] dS - \\ &\quad \frac{n}{32\pi^2} \left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}} \ln \left(\frac{(4S)^{\frac{1}{n-1}}}{2\pi l} \right) dQ, \\ d\mathcal{M} &= \frac{r_+}{4\pi} \left(\frac{n}{l^2} - \frac{2^{n/2} q^n}{r_+^n} \right) dS + \left(-q \ln \left(\frac{r_+}{l} \right) \right) dQ. \end{aligned}$$

Then we should differentiate it to obtain

$$\begin{aligned} d\mathcal{M} &= \frac{n}{32\pi^2} \left[\frac{4^{\frac{n}{n-1}} S^{\frac{1}{n-1}}}{l^2} - \frac{\left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}}}{nS} \right] dS - \\ &\quad \frac{n}{32\pi^2} \left(\frac{2^7 \pi^2 Q^2}{n^2} \right)^{\frac{n}{2(n-1)}} \ln \left(\frac{(4S)^{\frac{1}{n-1}}}{2\pi l} \right) dQ. \end{aligned}$$

Here, we should replace Q and S from Eqs. (17) and (18), and rewrite $d\mathcal{M}$

$$d\mathcal{M} = \frac{r_+}{4\pi} \left(\frac{n}{l^2} - \frac{2^{n/2} q^n}{r_+^n} \right) dS + \left(-q \ln \left(\frac{r_+}{l} \right) \right) dQ.$$

It is completely clear that the coefficients of dS and dQ are temperature T and electric potential U , respectively and so we have

$$d\mathcal{M} = TdS + UdQ.$$

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